

Simple protocol for generating W states in resonator-based quantum computing architectures

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We describe a simple, practical scheme for generating multi-qubit W states in resonator-based architectures, in which N Josephson phase qubits are capacitively coupled to a common resonator bus. The entire control sequence consists of three pulses: a local Rabi pulse that excites a single qubit in the circuit; a coupling pulse that transfers the qubit excitation to the resonator bus; and the main, entangling operation that simultaneously couples the bus to all N qubits. If the qubit-resonator coupling strength g is much smaller than the qubit energy splitting E_{10} , the system initially excited into the near-degenerate single-excitation subspace stays within that subspace, while smoothly evolving toward the fully uniform W state superposition. The duration of the final entangling operation is found to *decrease* with the total number of the qubits according to $t = \pi/(2g\sqrt{N})$, in agreement with some of the previously proposed cavity QED W state generation schemes.

I. DESCRIPTION OF THE W PROTOCOL

Our control sequence consists of the following three steps:

1. First, the initial local Rabi pulse is applied to one of the qubits in the circuit, bringing the qubit from its ground state $|0\rangle$ to the excited state $|1\rangle$,

$$|00\dots 000_r\rangle \rightarrow |00\dots 010_r\rangle. \quad (1)$$

2. The corresponding qubit-resonator coupling is then turned on, which transfers the excitation state from the qubit to the bus,

$$|00\dots 010_r\rangle \rightarrow |00\dots 001_r\rangle. \quad (2)$$

3. The second entangling pulse is applied, which couples the bus to all the qubits in the system. If the coupling g is much smaller than the qubit energy splitting E_{10} , the system initially prepared in the single-excitation subspace stays within that subspace while smoothly evolving toward the multi-qubit, fully uniform W-state superposition as follows,

$$|00\dots 001_r\rangle \rightarrow \left[\frac{|00\dots 01\rangle + |00\dots 10\rangle + \dots + |10\dots 00\rangle}{\sqrt{N}} \right] \otimes |0_r\rangle. \quad (3)$$

This last step is similar to the W state generation scheme proposed for cavity QED in Ref. [1]. In the terminology of reference [2], our resonator bus plays the role of the entanglement mediator.

II. MATHEMATICAL PRELIMINARY

It is well-known how to perform the first two operations of the W sequence described above [3]. We can therefore assume that the circuit was initially prepared in the state $|00\dots 001_r\rangle$, with only the bus excited. By simultaneously turning on the N couplings, the W state of the N -qubit network can then be generated using a single entangling operation (cf. [1]).

In order to see how this works, we consider a formal problem of an “effective” Hamiltonian,

$$H^{(N+1)} = g \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad (4)$$

which operates within a certain $(N+1)$ -dimensional Hilbert space $\mathcal{H}^{(N+1)}$, whose physical significance will be clarified below. The spectrum of $H^{(N+1)}$ is found to be

$$E^{(N+1)} = \mp\sqrt{N}, 0, \dots, 0. \quad (5)$$

The corresponding eigenvector matrix $S^{(N+1)}$, which diagonalizes $H^{(N+1)}$ via $H_{\text{diag}}^{(N+1)} = S^{(N+1)\dagger} H^{(N+1)} S^{(N+1)}$, is given by

$$S^{(N+1)} \sim \begin{pmatrix} -\sqrt{N} & \sqrt{N} & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & -1 & -1 & \dots & -1 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (6)$$

where we left the columns of $S^{(N+1)}$ unnormalized for notational simplicity. Direct exponentiation then shows

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that the N -dimensional uniform superposition state in this “effective” $(N+1)$ -dimensional system can be generated via

$$\frac{1}{\sqrt{N}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} = ie^{-iH^{(N+1)}t^{(N)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \quad (7)$$

where

$$t^{(N)} \equiv \frac{\pi}{2g\sqrt{N}}. \quad (8)$$

III. W_N STATE GENERATION

Our W state generation scheme is based on the idea that the effective Hamiltonian $H^{(N+1)}$ considered above should be viewed as operating within the single-excitation subspace of a network consisting of N qubits coupled to a common resonator bus. The corresponding mapping (extended by linearity) between the “effective” Hilbert space $\mathcal{H}^{(N+1)}$ and the $(N+1)$ -dimensional single-excitation subspace of the system may be chosen to be

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \rightarrow |00\dots 001_r\rangle, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix} \rightarrow |00\dots 010_r\rangle, \dots \quad (9)$$

The uniform N -dimensional superposition state generated in accordance with Eq. (22) then corresponds to the W_N state of the N qubits attached to the common bus.

In order for this approach to work, the single-excitation subspace has to be well isolated from the rest of the system’s Hilbert space. This near-degeneracy condition is typically well satisfied in various superconducting qubit architectures whose couplings, $g \simeq 100$ MHz, are much smaller than the qubit and resonator level splittings of $E_{10} \simeq 10$ GHz.

Let us check that the Hamiltonian $H^{(N+1)}$ arises naturally within the single-excitation subspace of a capacitively coupled network consisting of superconducting phase qubits and a resonator bus. When projected into the computational subspace spanned by the eigenfunctions $|0\rangle, |1\rangle, |2\rangle$ of the individual Josephson phase qubits (as well as the resonator), the Hamiltonian of such a network is given by

$$H = \sum_{i=1}^N H_i + H_r + \sum_{i=1}^N g_{ir} p_i p_r, \quad (10)$$

where the index i numbers the qubits and r labels the bus, with

$$\begin{aligned} H_1 &= \begin{pmatrix} -E_{10} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & E_{10} - \Delta_1 \end{pmatrix}, \\ H_j &= \begin{pmatrix} -E_{10} & 0 & 0 \\ 0 & \epsilon_j & 0 \\ 0 & 0 & E_{10} + 2\epsilon_j - \Delta_j \end{pmatrix}, \quad j = 2, \dots, N, \\ H_r &= \begin{pmatrix} -E_r & 0 & 0 \\ 0 & \epsilon_r & 0 \\ 0 & 0 & E_r + 2\epsilon_r \end{pmatrix}. \end{aligned} \quad (11)$$

The generalized momenta p_i and p_r are given by

$$\begin{aligned} p_i &= \lambda_2 + b\lambda_5 + c\lambda_7 = i \begin{pmatrix} 0 & -1 & -b_i \\ 1 & 0 & -c_i \\ b_i & c_i & 0 \end{pmatrix}, \\ p_r &= \lambda_2 + \sqrt{2}\lambda_7 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \end{aligned} \quad (12)$$

where λ_k , $k = 1, 2, \dots, 8$, are the standard Gell-Mann generators of the Lie algebra $\mathfrak{su}(3)$. In the above, g_{ir} are the qubit-bus coupling constants, E_{10} is the energy splitting of the first (reference) qubit, ϵ_j, ϵ_r , $j = 2, 3, \dots, N$, are the energy shifts relative to the single-excitation energy of the first qubit, Δ_i , $i = 1, 2, \dots, N$, are the qubit anharmonicities, b_i and c_i are the off-diagonal matrix elements of the i th qubit momentum, and E_r is the resonator energy splitting. The $(N+1) \times (N+1)$ block of the Hamiltonian H acting within the $(N+1)$ -dimensional single-excitation subspace spanned by $|00\dots 001_r\rangle, |00\dots 010_r\rangle, \dots, |10\dots 000_r\rangle$, is then given by the real symmetric matrix,

$$H^{(N+1)} = \begin{pmatrix} \epsilon_r & g_{Nr} & g_{N-1r} & g_{N-2r} & \dots & g_{3r} & g_{2r} & g_{1r} \\ g_{Nr} & \epsilon_N & 0 & 0 & \dots & 0 & 0 & 0 \\ g_{N-1r} & 0 & \epsilon_{N-1} & 0 & \dots & 0 & 0 & 0 \\ g_{N-2r} & 0 & 0 & \epsilon_{N-2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots \\ g_{2r} & 0 & 0 & 0 & \dots & 0 & \epsilon_2 & 0 \\ g_{1r} & 0 & 0 & 0 & \dots & 0 & 0 & 0 \end{pmatrix}. \quad (13)$$

This immediately shows that the W_N state can be generated if we place all system elements on resonance with each other by choosing

$$\begin{aligned} \epsilon_2 = \epsilon_3 = \epsilon_4 = \dots = \epsilon_N = \epsilon_r &= 0, \\ g_{1r} = g_{2r} = g_{3r} = \dots = g_{Nr} &= g. \end{aligned} \quad (14)$$

IV. NUMERICAL SIMULATION RESULTS

We tested this scheme on an $N = 4$ qubit network with $g = 100$ MHz, $E_{10} = E_r = 10$ GHz, $\Delta_j = 250$ MHz, and

$$p_i = i \begin{pmatrix} 0 & -1 & -0.08 \\ 1 & 0 & -1.43 \\ 0.08 & 1.43 & 0 \end{pmatrix}. \quad (15)$$

Assuming the system starts in the excited state $|00\dots 001_r\rangle$, the simulated final state of the system is found to be

$$|W_N\rangle_{\text{sim}} = \begin{pmatrix} -0.0003i \\ 0.4999 \\ 0.4999 \\ 0.4999 \\ 0.4999 \end{pmatrix}, \quad (16)$$

with the corresponding entangling time being $t = 1.2500$ ns. Ignoring the decoherence effects, the intrinsic fidelity [4] of the found state $|W_N\rangle_{\text{sim}}$ relative to the ideal W_N state, is

$$\mathcal{F}_N \equiv |\langle W_N | W_N \rangle_{\text{sim}}|^2 = 0.9994. \quad (17)$$

V. W_{N+1} STATE GENERATION

In a similar manner, the W_{N+1} state can also be generated, in which the resonator is maximally entangled with the qubits. This corresponds to the sequence of operations

$$\begin{aligned} |00\dots 000_r\rangle &\rightarrow |00\dots 010_r\rangle \rightarrow \\ &\frac{|00\dots 001_r\rangle + |00\dots 010_r\rangle + \dots + |10\dots 000_r\rangle}{\sqrt{N+1}}. \end{aligned} \quad (18)$$

In this scenario, we take full advantage of qubit tunability to construct the “effective” single-excitation Hamiltonian of the form

$$H^{(N+1)} = g \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (19)$$

Its spectrum and the diagonalizing transformation matrix $S^{(N+1)}$ (here shown unnormalized) are

$$E^{(N+1)} = 1 \mp \sqrt{N+1}, 0, \dots, 0, \quad (20)$$

and

$$\begin{pmatrix} 1 - \sqrt{N+1} & 1 + \sqrt{N+1} & 0 & 0 & 0 & \dots & 0 \\ 1 & 1 & -1 & -1 & -1 & \dots & -1 \\ 1 & 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & 1 & 0 & \dots & 0 \\ 1 & 1 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \quad (21)$$

respectively. The corresponding W state is generated via

$$\frac{1}{\sqrt{N+1}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ \dots \\ 1 \end{pmatrix} = ie^{i\alpha^{(N+1)}} e^{-iH^{(N+1)}t^{(N+1)}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \dots \\ 0 \end{pmatrix}, \quad (22)$$

where

$$\alpha^{(N+1)} = \frac{\pi}{2\sqrt{N+1}}, \quad t^{(N+1)} \equiv \frac{\pi}{2g\sqrt{N+1}}, \quad (23)$$

provided we detune the qubits from the resonator by $2g$,

$$\begin{aligned} \epsilon_2 = \epsilon_3 = \epsilon_4 = \dots = \epsilon_N = 0, \quad \epsilon_r = 2g, \\ g_{1r} = g_{2r} = g_{3r} = \dots = g_{Nr} = g. \end{aligned} \quad (24)$$

The duration of the entangling pulse is now $t = 1.1180$ ns, with the simulated single-excitation final state of the network being

$$|W_{N+1}\rangle_{\text{sim}} = \begin{pmatrix} 0.4471 - 0.0003i \\ 0.4472 \\ 0.4472 \\ 0.4472 \\ 0.4472 \end{pmatrix}, \quad (25)$$

with fidelity

$$\mathcal{F}_{N+1} = 0.9997. \quad (26)$$

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